

NOTE

THE SIZE OF 3-CROSS-FREE FAMILIES*

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We give a short and simple proof for the theorem that the size of a 3-cross-free family is linear in the size of the groundset. A family is 3-cross-free if it has no 3 pairwise crossing members.

1. Introduction

We shall prove that if $\mathcal{F} \subset 2^V$ is a 3-cross-free family then $|\mathcal{F}| \leq 10|V|$. We call $\mathcal{F} \subset 2^V$ to be *k-cross-free* if \mathcal{F} has no k pairwise crossing members. Sets $A, B \subset V$ are *crossing* if none of $A \cap B, A \setminus B, B \setminus A$ and $V \setminus (A \cup B)$ is empty.

It was conjectured by Karzanov and Lomonosov that $|\mathcal{F}| = O(kn)$ if \mathcal{F} is k -cross-free and $|V| = n$. For $k = 2$, this is trivial from the well-known tree representation of laminar families. In [7], Pevzner gave a quite complicated and lengthy proof for the case $k = 3$. In Section 2, we present a direct and easy proof for this result. Actually, we prove a slightly more general theorem than the one indicated above. We call a family $\mathcal{F} \subset 2^V$ *weakly k-cross-free with respect to* $a \in V$, if for every $b \in V \setminus \{a\}$ there are no k pairwise crossing members of \mathcal{F} separating a from b . We say that X *separates* a from b if it contains exactly one of them. We call a family *weakly k-cross-free* if it is weakly k -cross-free with respect to some element a of V . In Section 2, we will show that the size of a weakly 3-cross-free family is at most $10n$.

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As far as we know, the conjecture of Karzanov and Lomonosov is still open for $k > 3$ and the best known bound is $|\mathcal{F}| = O(kn \log n)$ due to Lomonosov. However, recently there has been some new results obtained by Dress *et al.* [2,1], where all maximum-size 3-cross-free families are described and the conjecture is proved for so-called cyclic 4-cross-free families. There, *cyclic* means that there is a cyclic order on V such that any element of \mathcal{F} is an interval in it. They also show that contrary to some naive expectations, maximum-size k -cross-free families are not cyclic for $k=4$.

The background of the investigation of 3-cross-free families is the so called locking theorem of Karzanov and Lomonosov [4,5] (see also [3]): a family $\mathcal{F} \subset 2^V$ is lockable if and only if \mathcal{F} is 3-cross-free. Family \mathcal{F} is called *lockable* if whenever $G=(W,E)$ is an undirected graph with $V \subset W$ then there exists a fractional path-packing (i.e. a multiflow) f in G such that every $X \in \mathcal{F}$ is locked in G by f . Subset X of V is *locked in G by f* if the total f -value of paths between X and $V \setminus X$ equals the minimum size of the edge-cuts of G separating X from $V \setminus X$. The following stronger version was also proved in [4,5] (for a shorter proof see [6]): \mathcal{F} is 3-cross-free if and only if for any $G=(W,E)$ inner Eulerian graph (that is, $V \subseteq W$ and the degrees of all vertices of $W \setminus V$ are even) there is a collection \mathcal{P} of edge-disjoint paths of G in such a way that for any $X \in \mathcal{F}$, \mathcal{P} contains maximum number of paths connecting X to $V \setminus X$.

2. Weakly 3-cross-free families

In this section we prove our result. Throughout we use the following notation:

$$\begin{aligned}\mathcal{F}/v &:= \{X \setminus \{v\} : X \in \mathcal{F}\} \\ \mathcal{F}(v) &:= \{X : v \in X \in \mathcal{F} \ni X \setminus \{v\}\}\end{aligned}$$

Theorem 1. *Let $|V|=n \in \mathbb{N}$ and let $\mathcal{F} \subset 2^V$ be a weakly 3-cross-free family. Then $|\mathcal{F}| \leq 10n$.*

Proof. Assume to the contrary that \mathcal{F} is a counterexample with $|V|$ minimal, that is, $|\mathcal{F}| > 10n$ and \mathcal{F} is weakly 3-cross-free with respect to a . Let us define $\mathcal{F}' := \{X \in \mathcal{F} : a \notin X\} \cup \{V \setminus X : a \in X \in \mathcal{F}\}$. Clearly, $|\mathcal{F}'| > 5n$ with the property that

(1) *if $X, Y, Z \in \mathcal{F}'$ with $X \cap Y \cap Z \neq \emptyset$ then X, Y, Z cannot pairwise cross.*

Next we prove:

(2) *For each $x \in V \setminus a$, there exist $A_x, B_x \in \mathcal{F}'(x)$ such that $B_x \neq A_x \subset B_x$ and $|A_x| \geq 3$.*

If $\{x\} \neq P \subset Q \subset R$ is a chain of three different elements from $\mathcal{F}'(x)$, then $A_x = Q$, $B_x = R$ suffices. Otherwise each element of $\mathcal{F}'(x) \setminus \{x\}$ is either inclusionwise minimal or maximal. By (1), we see that $\mathcal{F}'(x) \setminus \{x\}$ contains

at most two maxima and at most two minima, hence altogether $|\mathcal{F}'(x)| \leq 5$. As $|\mathcal{F}(x)| \leq 2|\mathcal{F}'(x)|$, we get that $|\mathcal{F}/x| = |\mathcal{F}| - |\mathcal{F}(x)| > 10|V| - 2|\mathcal{F}'(x)| \geq 10|V \setminus \{x\}|$. This contradicts to the minimality assumption as \mathcal{F}/x is also a weakly 3-cross-free family with respect to a . This proves (2).

Choose $x \in V \setminus a$, such that $|B_x|$ is as small as possible. Let $y, z \in A_x \setminus \{x\}$ be different elements. Observe that $y \in A_x \cap (B_x \setminus \{x\}) \cap B_y$ and that A_x crosses $B_x \setminus \{x\}$. By the choice of x , $|B_y| \geq |B_x|$, hence B_y must contain A_x or $B_x \setminus \{x\}$ by (1). In particular, $z \in A_x \setminus \{x\} \subset B_y$ holds. Then $z \in A_x \cap (B_x \setminus \{x\}) \cap (B_y \setminus \{y\})$, and these three sets of \mathcal{F}' pairwise cross, contradicting (1). ■

3. Conclusions

As indicated, Karzanov's conjecture about the linear size of k -cross-free families is still open for $k > 3$. However Lomonosov's argument is also valid in our weakly k -cross-free setting. Indeed, let $\mathcal{F}^i := \{X \in \mathcal{F}' : |X| = i\}$ for $i = 0, 1, \dots, n$, where \mathcal{F}' is defined as in the proof of Theorem 1. Clearly, for every $v \in V \setminus a$ there are less than k sets in \mathcal{F}^i covering v , hence $|\mathcal{F}| \leq 2|\mathcal{F}'| = 2 \sum_{i=0}^n |\mathcal{F}^i| < 2(1 + \sum_{i=1}^n \frac{kn}{i}) = O(kn \log n)$.

Pevzner published a paper about the linear size of 3-cross-free families [7], in which he explores important properties of k -cross-free and 3-cross-free families. Although the proof is not easy to read, he had some interesting remarks that are worth citing. In our terminology his question is the following:

Is it true that any k -cross-free family on n elements can be decomposed into r $(k - 1)$ -cross-free families (r is independent of $n, k > 3$)?

He also observes:

It is possible to show that for $k = 3$ the answer to the above problem is negative (an example of an r -indecomposable 3-cross-free family is a family of stars in a graph without triangles with a chromatic number exceeding r).

It is interesting to see that the answer to Pevzner's question is negative even for all k if we ask it for families that are weakly k -cross-free with respect to a fixed point of the groundset: Let $[n] := \{i \in \mathbb{N} : 1 \leq i \leq n\}$; $\binom{[n]}{k} := \{X \subset [n] : |X| = k\}$ and define $\mathcal{F}([n], k) := \{X \in \binom{[n]}{k} : i \in X \Rightarrow i \in [n] \subset 2^{\binom{[n]}{k}}\}$. Although for $k \geq 2, n \geq 4$ and $X \in \binom{[n]}{k}$ family $\mathcal{F}([n], k)_X := \{F \in \mathcal{F}([n], k) : X \notin F\}$ consists of pairwise crossing sets, it is already weakly $(k + 1)$ -cross-free with respect to X . Moreover, any k elements of $\mathcal{F}([n], k)_X$ separate X from another element Y of $\binom{[n]}{k}$, hence for $n \geq (c + 1) \cdot k$ it is not possible to partition $\mathcal{F}([n], k)_X$ into c families that are all weakly k -cross-free with respect to X .

Our last remark is that Theorem 1 is not very far from the best possible bound: notice that $\mathcal{F}[n, k] := \{i + [j], [i] + j, [n] \setminus (i + [j]), [n] \setminus ([i] + j) : i + 1 \in [k], j \in [n - i]\} \subset 2^{[n]}$ (where $a + [b] := [a + b] \setminus [a]$) is a k -cross-free family with roughly $4(k - 1)n$ members. In particular, there is a 3-cross-free family $\mathcal{F}[n, 3]$ with roughly $8n$ members.

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